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SOME COMMENTS ON ELASTIC–PLASTIC ANALYSIS

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Abstract—Definitions of plastic and elastic deformation are discussed which are directly related to the physical characteristics of the phenomena and which provide essential uncoupling of elastic and plastic variables. Using finite deformation kinematics and appropriate strain-rate variables in order to carry out total incremental theory, analysis is presented concerning the fact that the total strain rate is not equal to the sum of elastic and plastic rates as is universally assumed. The consequences of this circumstance are examined.

It is shown that a conflicting approach to the finite-deformation kinematics of elastic–plastic theory by Nemat-Nasser[4], which generates summable elastic and plastic strain rates, exhibits undesirable features and in particular the associated kinetics does not permit completion of the objective of developing an elastic–plastic constitutive relation on this basis.

1. INTRODUCTION

Recently two articles[1, 2] have been issued presenting incremental, finite-deformation, elastic–plastic analysis based on finite-deformation nonlinear kinematics according to which the total strain rate is *not* equal to the sum of the elastic and plastic strain rates. The theory presented is the incremental version of that published in 1969[3], which was expressed in terms of the incremental plastic strain relation with the stress given as the usual finite-deformation nonlinear function of the elastic deformation gradient. The previous failure to express this theory in incremental total-strain form has hampered the application of it. The recent articles mentioned were stimulated by a paper by Nemat-Nasser[4] which emphasized and elaborated on the validity of the summability of the elastic and plastic strain rates at finite deformation, an assumption which forms the basis of virtually all analytical developments and computer codes for elastic–plastic analysis including those specifically devised for finite-deformation problems. Prior to publication of [4], we decided that it would be more useful to others working in this area if the disagreement concerning formulation of elastic–plastic theory appeared in print rather than us attempting to resolve it at the refereeing level. The present paper is an attempt to elucidate aspects of the problem.

From discussions of the theory with others, it became clear that the mathematical representation of the physical entities involved generates most controversy. As Eddington said in describing the solution of a physical problem[5], "the initial formulation of the problem was the most difficult part, as it was necessary to use one's brains all the time; afterwards one could use mathematics instead!" It is of overriding importance to understand the physical characteristics of the phenomenon before attempting the mathematical representation.

Difficulty arises in the formulation of elastic–plastic theory because the laws governing elastic and plastic response have quite different structures. However, up to moderate strains (of the order 30%[6]) the elastic and plastic properties can be essentially uncoupled if expressed appropriately. It was shown by Howard and Smith[7] that by taking the macroscopically unstressed state as the reference state, the elastic response to applied stresses of polycrystalline metals is not appreciably influenced by prior plastic flow. The term "macroscopically unstressed" is used since, after plastic flow, stresses occur on the micro-scale localized around the generated dislocations. The essentially invariant elastic properties arise because the dislocations and other lattice defects caused by plastic flow, leave only a very small proportion of atoms not in regular crystal lattice positions and it is the deformation of the crystal lattice that determines the elastic constants. Stress analysts are of course aware that the elastic constants of a material are not appreciably influenced by the manufacturing process which produced the component under study, although the process might have involved plastic forming. Nor are the plastic characteristics of a body appreciably influenced by elastic deformation as exhibited, e.g. by Bridgman who observed that the stress–strain curve in tension was not appreciably modified by superposed hydrostatic compression[8].

Plastic deformation is that remaining when the stress in a body subjected to elastic–plastic deformation is reduced to zero and thus also is the elastic strain. Physically plastic strain is caused by the migration of dislocations and other defects through the crystal lattice, so that there is a direct correspondence between the defect pattern and history in the otherwise unstrained lattice, and the plastic strain.

The onset of plasticity in a tension test is detected by loading the specimen and then removing the load to check after what stress magnitude residual strain first appears. Thereafter the plastic strain can be measured by reducing the stress to zero after loading and measuring the residual strain as illustrated in Fig. 1. If the destressing from A along AB is elastic, the residual strain OB, which is the plastic strain at B, is also the plastic strain in the stressed configuration at A since no change in plastic strain has occurred during the elastic unloading (dislocations have remained stationary).

For continued measurements of plastic strain along the stress–strain curve as the stress increases, it is awkward to unload the specimen for each plastic strain measurement, particularly since at large stresses the unloading may not be purely elastic. Thus it is common to take measurements closer to the elastic–plastic stress–strain curve by adding a small increment of stress $\Delta\sigma$ along CD, Fig. 1 and then removing it along DF. After the stress is thus cycled back to $\sigma(F) = \sigma(C)$, a residual increment of strain $CF = \Delta\epsilon^{\text{res}}$ has been generated, and a reversible increment $FG = \Delta\epsilon^{\text{rev}}$. $\Delta\epsilon^{\text{res}}$ is commonly considered to be the increment of plastic strain $\Delta\epsilon^p$ generated along CD

$$\Delta\epsilon^p = \Delta\epsilon^{\text{res}} \tag{1.1}$$

and $\Delta\epsilon^{\text{rev}}$ the increment of elastic strain

$$\Delta\epsilon^e = \Delta\epsilon^{\text{rev}}. \tag{1.2}$$

Since the strain increments are considered infinitesimal, these definitions clearly imply that

$$\Delta\epsilon = \Delta\epsilon^e + \Delta\epsilon^p. \tag{1.3}$$

As the stress–strain curve is traversed the total elastic and plastic strains accumulated are obtained by adding the successive increments of strain.

In using this incremental approach to the measurement of plastic strain along a stress–strain curve, the increments of plastic strain are measured on a body which is subjected to the continuously increasing stress, σ, so that the specimen is continuously deformed by the elastic strain σ/E, where E is Young's modulus. The question then arises whether this elastic strain

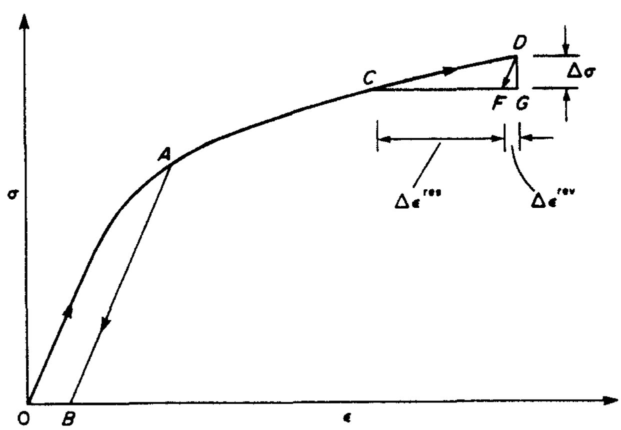

Fig. 1. Tension test.

modifies the plastic strain measurement since the plastic strain increments no longer occur in an elastically unstrained element and so will not correspond directly to the dislocation pattern in the elastically unstrained lattice which constitutes the definition of plastic strain.

It was the concept of such a coupling between elastic strain and residual strain increments (assumed to be plastic strain increments) that led us to develop the nonlinear finite-deformation kinematics used in [1–3]. A grossly exaggerated model which illustrates qualitatively the influence that such coupling might have is shown in Fig. 2. Figure 2(a) indicates the shear strain $\Delta\gamma^p$ generated by passage of an edge dislocation across an elastically unstrained element. Suppose now tha the element is stretched elastically vertically by a stretch ratio 2 as shown in Fig. 2(b). Traverse by the same lattice dislocation will produce a plastic shear strain increment $\Delta\underline{\gamma}^p = \Delta\gamma^p/2$. Clearly the plastic strain increment measured on the elastically deformed specimen will not have the desired direct correspondence with the dislocation pattern uncoupled from the elastic strain, and hence the measured plastic strain will not be in accordance with the physical theory of plasticity. Moreover, since the elastic strain may change during the period of plastic flow, addition of measured plastic strain increments will not yield a sensible plastic strain (or generalized plastic strain, $\bar{\epsilon}^p$) variable since the significance for plastic flow will vary for each increment depending on the stage in the test at which it occurred. The finite-deformation kinematic analysis [1, 2], discussed in the following section, shows that this anomaly is much less significant than the related influence of rotation during the loading. Both anomalies can be eliminated from the analysis by appropriate selection of plastic and elastic deformation variables and by incorporating the resulting nonlinear kinematics as discussed in the following section.

Another coupling effect which can also be avoided by appropriate selection of variables is the effect of plastic deformation on the elastic response. Figure 3 illustrates load deformation relations in a tensile test of the type presented by Howard and Smith [7] and by Dalby [9]. This is of course equivalent to a nominal stress-nominal strain plot. A loading test involving elastic–plastic deformation followed by elastic unloading reveals that the gradient of the elastic unloading line AB is less steep than the initial elastic loading line OY and that this effect increases with plastic flow. However, by taking account of the change in cross-sectional area of the specimen and the extension, mainly due to plastic flow, by calculating the true stress and elastic strain increment using the current configuration as reference, the elastic lines yield an essentially constant slope equal to the Young's modulus. This result would have become obvious if true-stress and natural or logarithmic strain had been plotted, but this approach cannot conveniently be applied when general deformation including rotation is being analysed. If the unstressed configuration, corresponding to B in Fig. 3, is used to express elastic strain increments, corresponding to the line BA, elastic relations involving Cauchy or Kirchhoff stress ($\tau = (\rho_0/\rho)\sigma$) will be effectively independent of plastic strain for most metals subjected to moderate strains. This circumstance determines the selection of the unstressed state in general deformation as the reference state for expressing the elastic deformation gradient F^e in order to

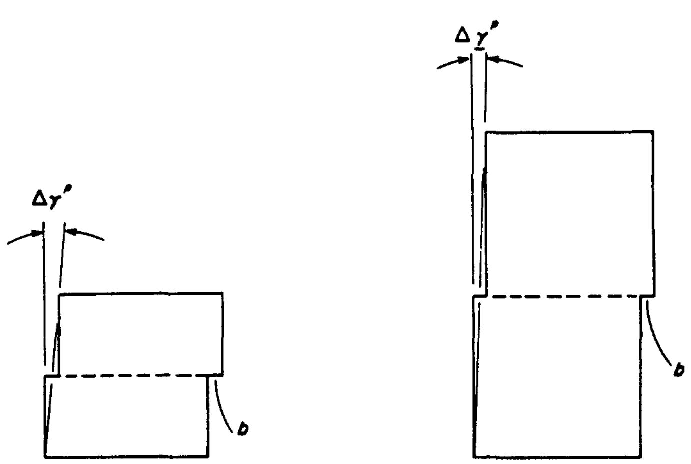

Fig. 2. Dislocation traverse in an elastically unstrained and strained element.

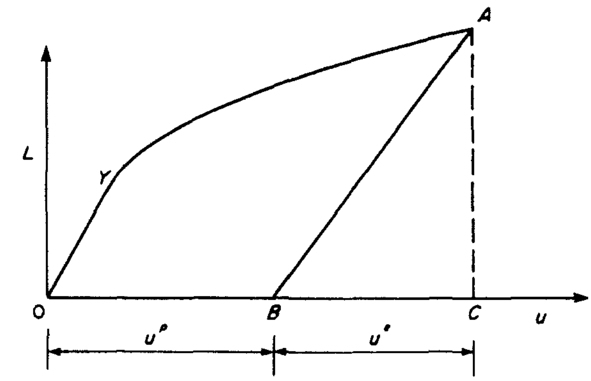

Fig. 3. Load extension curve in tension.

take maximum advantage of uncoupling the elastic and plastic relations in formulating elastic–plastic theory.

2. KINEMATICS

The physical concepts discussed in the introduction suggest analysis of the deformation based on a loading–unloading cycle of the type depicted in Fig. 3 for a tensile test. Figure 3, with modified notation, is repeated in Fig. 4 along with an analogous test involving general deformation. The kinematics is that developed in [1–3] but the notation of [4] is adopted to facilitate comments on that paper.

The configuration of the initially undeformed body is expressed by the particle coordinates labeled $\mathbf{X}$ in Fig. 4(b). The body is subjected to elastic–plastic deformation to the configuration $\mathbf{x}$ defined by the mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{2.1}$$

De-stressing yields the unstressed configuration defined by particle positions $\mathbf{p}$. The analogous situation for a tensile test is shown in Fig. 4(a) with the same labeling. The analysis of the general

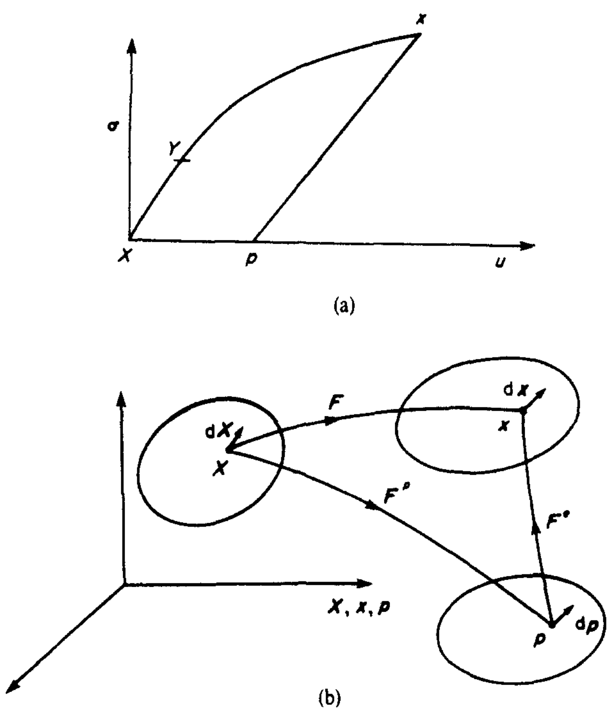

Fig. 4. Elastic–plastic deformation.

deformation is expressed in terms of the deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}; \quad F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{2.2}$$

for the total elastic–plastic deformation to the configuration $\mathbf{x}$. Destressing occurs from $\mathbf{x}$ to $\mathbf{p}$ and is considered to be elastic and so reversible. The stress and so the elastic strain is zero in the configuration $\mathbf{p}$, so that the elastic deformation in the configuration $\mathbf{x}$ is given by the deformation gradient $\mathbf{F}^e$.

$$\mathbf{F}^e = \frac{\partial \mathbf{x}}{\partial \mathbf{p}}; \quad F^e_{ij} = \frac{\partial x_i}{\partial p_j}. \tag{2.3}$$

Since the stress is zero at $\mathbf{p}$, the deformation is purely plastic and also constitutes the plastic deformation at $\mathbf{x}$ since no plastic flow occurs from $\mathbf{p}$ to $\mathbf{x}$ due to the reversibility. The plastic deformation at $\mathbf{p}$ *and* $\mathbf{x}$ is thus expressed by the deformation gradient $\mathbf{F}^p$

$$\mathbf{F}^p = \frac{\partial \mathbf{p}}{\partial \mathbf{X}}; \quad F^p_{ij} = \frac{\partial p_i}{\partial X_j}. \tag{2.4}$$

As has been explained in [1, 3, 4], because unloading following plastic flow usually leaves a distribution of residual stresses in a body, reducing the stress to zero demands a discontinuous or multi-valued map of $\mathbf{X}$ onto $\mathbf{p}$ between which configurations partial derivatives do not exist. $\mathbf{F}^p$ and $\mathbf{F}^e$ then become point functions of the particles $\mathbf{X}$ and not partial derivatives, without adding essential complexity to the theory.

We are concerned with analysing the elastic–plastic deformation from $\mathbf{X}$ to $\mathbf{x}$ which is usually a continuous differentiable mapping. The *concept* of destressing to the unstressed state $\mathbf{p}$ is needed to define the elastic and plastic deformations involved in the configuration $\mathbf{x}$ as already described. Thus the unstressed configuration $\mathbf{p}$ is not usually physically achieved but simply provides a construction by means of which the elastic and plastic deformations can be expressed, a guide to the selection of appropriate variables. Of course, in principle, state $\mathbf{p}$ could be achieved physically by having $\mathbf{x}$ follow the destressing path until it coincides with $\mathbf{p}$.

Given the configurations $\mathbf{X}$, $\mathbf{x}$ and $\mathbf{p}$, the total mapping $\mathbf{X} \to \mathbf{x}$ can be expressed mathematically, by the sequence of mappings $\mathbf{X} \to \mathbf{p}$ and $\mathbf{p} \to \mathbf{x}$ (even though the deformation $\mathbf{X} \to \mathbf{p}$, at zero stress, is not possible physically), and the chain rule then yields

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{2.5}$$

which expresses the total deformation in terms of the elastic and plastic components. Note that these quantities can all in principle be measured so that the kinematics of elastic–plastic deformation is based on variables directly associated with physically achievable configurations of the system. Note also, that the order of the usually non-commutative product (2.5) arises because the elastic path $\mathbf{p} \to \mathbf{x}$ does not intersect $\mathbf{X}$ and is not related to whether elastic or plastic deformation occurs first. We are concerned with the elastic–plastic deformation $\mathbf{X} \to \mathbf{x}$ along which they usually are occurring simultaneously.

By substituting for $\mathbf{F}$ from (2.5) into the Lagrange strain $\mathbf{E}$

$$\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2 \tag{2.6}$$

we obtain:

$$\mathbf{E} = \mathbf{F}^{p^{-1}}\mathbf{E}^e\mathbf{F}^p + \mathbf{E}^p \tag{2.7}$$

where $\mathbf{E}^e$ and $\mathbf{E}^p$ are the elastic and plastic finite strains from their respective reference states

$$\mathbf{E}^e = (\mathbf{F}^{e^T}\mathbf{F}^e - \mathbf{I})/2; \quad \mathbf{E}^p = (\mathbf{F}^{p^T}\mathbf{F}^p - \mathbf{I})/2. \tag{2.8}$$

Since $\mathbf{F}^p$ can be far removed from the unit matrix, it is clear from (2.7), that summability of elastic and plastic strains as defined here has no validity.

The particle velocity in the state **x** is given by

$$\mathbf{v} = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\mathbf{X}} \tag{2.9}$$

and so the velocity gradient **L** in **x** by

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

where the superposed dot denotes material derivative, time differentiation at fixed **X**. Substitution from (2.5) then gives

$$\mathbf{L} = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} + \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\mathbf{F}^{e^{-1}}. \tag{2.10}$$

As was pointed out in [3], the configuration in the neighborhood of each material element in the unstressed state **p** is unique only to within an arbitrary rigid-body rotation. Such rotation does not change the strain so that the stress remains zero and the lack of continuity of the mapping, already referred to, permits the rotation to be chosen independently for each material element. It is convenient analytically and provides the simplest definition of unloading, to select the rotation so that, for each element, unloading from **x** occurs without rotation. $\mathbf{F}^e$ is then symmetric and is written:

$$\mathbf{F}^e = \mathbf{V}^e; \quad \mathbf{V}^e = \mathbf{V}^{e^T}. \tag{2.11}$$

Equation (2.10) then becomes

$$\mathbf{L} = \dot{\mathbf{V}}^e\mathbf{V}^{e^{-1}} + \mathbf{V}^e\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\mathbf{V}^{e^{-1}}. \tag{2.12}$$

The symmetric part, **D**, variously termed the rate of deformation, stretching tensor or velocity strain expresses the rate of strain about the current configuration and the anti-symmetric part gives the spin W, thus

$$\mathbf{D} = \mathbf{D}^e + [\mathbf{V}^e(\mathbf{D}^p + \mathbf{W}^p)\mathbf{V}^{e^{-1}}]_S = \mathbf{D}^e + \mathbf{V}^e\mathbf{D}^p\mathbf{V}^{e^{-1}}|_S + \mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}}|_S \tag{2.13}$$

$$\mathbf{W} = \mathbf{W}^e + [\mathbf{V}^e(\mathbf{D}^p + \mathbf{W}^p)\mathbf{V}^{e^{-1}}]_A = \mathbf{W}^e + \mathbf{V}^e\mathbf{D}^p\mathbf{V}^{e^{-1}}|_A + \mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}}|_A \tag{2.14}$$

where $\mathbf{D}^e$, $\mathbf{W}^e$, $\mathbf{D}^p$, $\mathbf{W}^p$ are the symmetric and anti-symmetric parts of $\dot{\mathbf{V}}^e\mathbf{V}^{e^{-1}}$ and $\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}$ respectively and subscripts S and A denote the symmetric and anti-symmetric parts respectively.

In spite of the discontinuous nature of the mapping $\mathbf{X}\rightarrow\mathbf{p}$ and the consequent lack of a global velocity gradient in **p**, $\mathbf{D}^p$ expresses the rate of plastic strain taking place in the currently plastically deforming configuration **p** since it expresses the rate of strain in each infinitesimal material element evaluated by means of the fixed reference state **X**.

The justification that $\mathbf{D}^e$ expresses the rate of elastic strain, associated with $\dot{\mathbf{F}}^e$, taking place in the currently deforming configuration **x**, requires more careful consideration since the zero-elastic-strain reference state **p** is continuously deforming plastically while constituting the unstressed reference state for expressing the finite-deformation elastic law. To simplify the circumstances we will first consider the conditions shown in the Introduction to be valid for most polycrystalline metals at moderate plastic strains, that the elastic law based on the unstressed reference state is isotropic and not appreciably changed by plastic flow. The elastic constitutive relation then determines the stress in a material element as a function of the current deformation from $\mathbf{p}\rightarrow\mathbf{x}$ at that element which is not affected by the plastic flow taking place. For example the Kirchhoff stress $\boldsymbol{\tau}(\mathbf{X}, t)$ is given by the objective constitutive relation

$$\boldsymbol{\tau}(\mathbf{X}, t) = 2\mathbf{F}^e(\mathbf{X}, t)\frac{\partial\psi}{\partial\mathbf{C}^e(\mathbf{X}, t)}\mathbf{F}^e(\mathbf{X}, t)^T; \quad \mathbf{C}^e(\mathbf{X}, t) = \mathbf{F}^e(\mathbf{X}, t)^T\mathbf{F}^e(\mathbf{X}, t) \tag{2.15}$$

where $\psi(\mathbf{C}^e, \theta)$ is the Helmholtz free energy per unit initial volume, an isotropic scalar function of C^e. The relationship between stress rate and strain rate will thus be obtained by differentiating (2.15) and will yield results identical with those for purely elastic deformation from a fixed reference state $\bar{X}$ and the same history of $F^e(\bar{X}, t)$ and $\tau(\bar{X}, t)$ as occurs in the elastic–plastic problem for a particular element in the neighborhood of X. For simplicity, homogeneous *purely–elastic* deformation could be considered corresponding to the elastic deformation in each material element in the body deforming elastically–plastically. In the purely elastic case a global velocity gradient will exist given by

$$\mathbf{L}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} \tag{2.16}$$

and the rate of elastic strain in the current configuration, $\mathbf{D}^e$, will be defined unambiguously and will thus also be appropriate for the elastic–plastic case in which the zero-stress reference state is continuously deforming.

In the case of an initially elastically anisotropic material (or one in which elastic anisotropy is induced by plastic flow), while rotation of the unstressed configuration p will leave it unstressed, it will modify the elastic constitutive equations using fixed axes. This is a special case of more general modification of the elastic law by plastic flow. Equation (2.15) will still apply with the free energy ψ not being an isotropic function of $\mathbf{C}^e$ and hence depending on the rotation of p. Thus the expression for the stress rate will involve not only $\dot{\mathbf{F}}^e$ but also terms associated with the deformation (including rotation) of p. Since the formulation of such coupling between the elastic law and plastic flow is not now established, we will restrict our consideration to the isotropic case.

$\mathbf{F}^p$ and $\mathbf{F}^e$ were chosen to model the physical characteristics of plasticity and elasticity and to minimize the coupling effects between them, and $\mathbf{D}^p$ and $\mathbf{D}^e$ were derived from them. Equation (2.13) shows that for this formulation

$$\mathbf{D} \neq \mathbf{D}^e + \mathbf{D}^p \tag{2.17}$$

in contradiction to the usual assumption in current common formulations of elastic–plastic theory.

Since the elastic strain in metals is usually small, $\sim 10^{-3}$, $\mathbf{V}^e = \mathbf{I} + \boldsymbol{\delta}$ where $\boldsymbol{\delta} \sim 10^{-3}$, so that by neglecting $\boldsymbol{\delta}$ compared with I, (2.13) is approximated by

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \tag{2.18}$$

Thus to zero order in δ, summability of strain rates is achieved in contrast to the complete failure of the summability of strains (2.7).

Equation (2.13) for the total strain rate can be used to assess the method of measuring material characteristics by means of incremental loading for general deformation, corresponding to the relations (1.1)–(1.3) for the tension test. This common method of measuring elastic–plastic material characteristics has implications in the finite-element stress analysis of elastic–plastic deformation as discussed in the following section.

Multiplying (2.13) throughout by the time increment Δt changes all rate terms into increments. It was pointed out in [1] that, D^e, the first term of the right hand equality expression of (2.13) corresponds to $\Delta\epsilon^{rev}$ in (1.2), the rest to $\Delta\epsilon^{res}$ in (1.1). The last term in (2.13), $\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}}|_S$, was shown to express the change in strain caused by the sequential operation of the component deformations starting from the right hand one, i.e. first removing the elastic strain, then applying an increment of rotation and then re-applying the elastic strain. This combination results in a change in elastic strain caused by rotation of the material element while the stress σ is maintained during application of the stress increment $\Delta\sigma$. This component of the residual strain increment should clearly not be considered as part of the plastic strain. Its inclusion in (2.18), which is the general deformation analogue of (1.3) represents an anomaly in the common strain rate summability assumption or hypothesis which is not in accord with the nonlinear finite-deformation kinematics that results in (2.13). If no rotation occurs, as in the tension test, the spin term is zero and this covers the simplest experimental programs. The kinematics for cases when stress variation is restricted to principal directions fixed in the body for isotropic materials can be analyzed using logarithmic or natural strain and then summability of both strains and strain-rates results[10, 1]. The anomaly does arise in finite-element analysis of elastic–plastic deformation since in such problems non-homogeneous distributions of deformation will usually occur which are likely to involve rotation. This matter is touched upon in the following section.

It is, perhaps, appropriate to clarify an overgeneral statement made by me in[3] that, for finite deformation kinematics, summability of elastic and plastic strains cannot give the total strain. I had in mind the ideas presented in this section that the definitions of elastic and plastic strains were selected to optimize uncoupling of these phenomena and to best represent the physical processes. It was pointed out by Green and Naghdi[11] and by Skachenko and Sporykhin[12] that by utilizing convected coordinates based on the initial reference configuration and thus expressing strains between two configurations as the difference between the metric tensors of the convected coordinate net in those configurations, summability of strains does arise. In fact, Skachenko and Sporykhin point out that Sedov[13] had based his development of elastic–plastic theory at finite strain on this approach. My statement in [3] overlooked this approach since the convected coordinate net in the unstressed state, which provides the most convenient reference configuration for expressing the elastic law, changes with plastic flow and so introduces a coupling of the elastic constitutive relation with plastic strain. This involved a coupling between elasticity and plasticity of the type I was attempting to avoid and hence that formulation was overlooked, resulting in a misleading statement.

3. ELASTIC-PLASTIC ANALYSIS

A not uncommon reaction to the elastic–plastic theory developed in [2] has been the question "Why can't a satisfactory elastic–plastic theory be obtained by *defining* the plastic strain increment as the residual strain increment left after application and removal of a stress increment?", thus adopting the general deformation analogue of (1.1). The "plastic strain rate" would then comprise the last two terms of (2.13) which thus involves both $\mathbf{D}^p$ and $\mathbf{W}^p$. That this choice would lead to error in stress analysis, if the usual elastic–plastic constitutive relation is used, is made evident by examining the kinematics involved.

As pointed out in [1, 2], for isotropic elastic and plastic response, the second term of the right-hand equality in (2.13) reduces to $\mathbf{D}^p$ since all the matrices involved have the same principal directions parallel to those of the stress tensor, $\boldsymbol{\tau}$ and are therefore commutative in multiplication. The normality rule sets $\mathbf{D}^p$ normal to the yield surface and hence parallel to the stress tensor for the Mises yield condition. However, the last term in (2.13) is normal to the stress tensor as shown by examining the scalar product of the stress and strain-rate vectors in the nine-dimensional stress and strain-rate space.

Starting with the corresponding term in the velocity gradient vector, the scalar product of this with the stress $\boldsymbol{\tau}$ is

$$\operatorname{tr}(\boldsymbol{\tau}\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}}) = \operatorname{tr}(\mathbf{V}^{e^{-1}}\boldsymbol{\tau}\mathbf{V}^e\mathbf{W}^p) = \operatorname{tr}(\boldsymbol{\tau}\mathbf{W}^p) = 0. \tag{3.1}$$

since $\boldsymbol{\tau}$ is symmetric and $\mathbf{W}^p$ anti-symmetric. Now the desired scalar product is

$$\operatorname{tr}[\boldsymbol{\tau}(\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_S] = \operatorname{tr}\{\boldsymbol{\tau}[\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}} - (\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_A]\} \tag{3.2}$$

and

$$\operatorname{tr}[\boldsymbol{\tau}(\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_A] = 0 \tag{3.3}$$

since $\boldsymbol{\tau}$ is symmetric. Thus from (3.1), (3.2) and (3.3) the desired scalar product of the stress with the spin associated component of the strain rate is zero:

$$\operatorname{tr}[\boldsymbol{\tau}(\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_S] = 0. \tag{3.4}$$

Moreover $(\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_S$ is not zero when $\mathbf{W}^p$ is not zero since removing the elastic strain, rotating the body and then re-imposing the strain produces a change in strain unless $\mathbf{V}^e = \mathbf{I}$. Thus the vector $(\mathbf{V}^e\mathbf{W}^p\mathbf{V}^{e^{-1}})_S$ is normal to the stress vector and hence tangential to the yield surface. This result was already evident from the discussion of plastic rate of work in [3] which was shown not to be affected by the rotation. The residual strain-rate vector, the "plastic strain rate", will therefore not be normal to the yield surface.

Thus if the sum of the last two terms in the right hand equality in (2.13) (the residual strain rate) is considered to be the "plastic strain rate", and inserted into the usual velocity variational principle with the usual constitutive relation requiring normality of the plastic strain rate, the stress point on the yield surface will be selected by the variational principle such that the stress vector will be parallel to the "plastic strain rate" vector and hence will not correspond to the correct position on the yield surface, leading to an incorrect stress determination. It is thus important to ignore this spin associated component of the residual strain rate in defining the plastic strain rate so that the correct normality condition will be preserved.

In the development of the constitutive relation presented in [2], objectivity was established by considering the superposition of an arbitrary rigid body rotational motion on the currently deformed configuration $\mathbf{x}$ at each material point and by checking the indifference of the kinematic variables used. As was emphasized in the previous section, we were considering the elastic–plastic stressing process $\mathbf{X} \to \mathbf{x}(\mathbf{X}, t)$ in Fig. 4(b). The unstressed state $\mathbf{p}$ was envisaged and analysed in order to select convenient variables with which to express the theory, but was not actually achieved during the stressing so that it is not necessary to impose a test of objectivity on that configuration. For analytical convenience the elastic and the plastic deformations were delineated by considering the process of unstressing from the configuration $\mathbf{x}$ without rotation, so that in the test for objectivity the rigid body rotation $\mathbf{Q}(\mathbf{x}, t)$ superimposed on $\mathbf{x}$ was also considered to be superimposed on $\mathbf{p}$ in order to preserve the destressing without rotation.

Green and Naghdi[11] and Casey and Naghdi[14] have advocated "full invariance requirements" for elastic–plastic analysis which demand indifference of the kinematic variables under independent time dependent rigid body rotations in both the current configuration $\mathbf{x}$ and the unstressed state $\mathbf{p}$. Since, as has already been explained, the unstressed state, $\mathbf{p}$, has been used only as a thought experiment by which to devise convenient variables for expressing elastic–plastic theory, such a double objectivity requirement is not needed and may well unduly complicate the analysis to no purpose. Were a check of objectivity of the unstressed state $\mathbf{p}(\mathbf{X}, t_1)$ desired, the body could be elastically deformed from $\mathbf{x}(\mathbf{X}, t_1)$ to $\mathbf{x}(\mathbf{X}, t_2) = \mathbf{p}(\mathbf{X}, t_1)$, when the test with $\mathbf{Q}(\mathbf{x}, t)$ would check the objectivity of that configuration. Demanding simultaneous checks of objectivity of both $\mathbf{x}$ and $\mathbf{p}$ seems to me to demand two objectivity checks for the single configuration $\mathbf{x}$, which would be redundant. Note that only $\mathbf{F}$ and functions derived from it and not $\mathbf{F}^p$, appears in the variational principle developed for stress analysis[2], so that the configuration $\mathbf{p}$ does not appear explicitly in the solution for the velocity field.

The theory already presented was devised for materials which respond elastically during destressing from $\mathbf{x}$ to $\mathbf{p}$ in Fig. 4. Then the strain in the configuration $\mathbf{p}$, which is purely plastic since the stress and hence the elastic strain, is zero there, is also the plastic strain in the stressed configuration $\mathbf{x}$, since the deformation between $\mathbf{p}$ and $\mathbf{x}$ is purely elastic. However, the approach to the analysis adopted can still be utilized when the material exhibits a Bauschinger effect so that reverse plastic flow occurs before the unstressed state $\mathbf{p}$ is reached. In such cases the configuration $\mathbf{p}$ still involves purely plastic strain from the initial configuration $\mathbf{X}$, but does not then also express the plastic deformation in the configuration $\mathbf{x}$. However an elastic free-energy function ψ will govern the strain in the elastic range as the stress is reduced before reverse plastic flow occurs. It will be insensitive to the dislocation field present, since, as already explained, at moderate strains ψ is governed mainly by the crystal lattice. The fact that dislocations are pinned against imperfections or grain boundaries by the prior plastic flow and retreat as the mutual repulsion between like dislocations takes precedence over the decreasing pinning stress, will not greatly affect the free energy function but mainly the range of stress over which it governs the deformation. That range is determined by the current yield condition. The elastic unloading law can thus be prescribed in terms of the free energy function, but the deformation at zero stress based on it will determine only a pseuso plastic strain corresponding to the state $\mathbf{x}$ which cannot be achieved by destressing from the most recent state at which plastic flow ceased. This non-achievable unstressed state provides a convenient vehicle for defining the elastic and plastic strains in the configuration $\mathbf{x}$, though these measures are not amendable to direct measurement and are not needed to prescribe the stress deformation relation in the elastic range itself.

It was pointed out in the Discussion Section of [2] that the usual, current, elastic–plastic, finite-deformation, computer code is based on the linear kinematics of the summation of elastic and plastic strain rates to give the total strain rate, the incremental law for the plastic strain rate and the derivative of Hooke's law for the elastic strain rate. This combination yields a total strain-rate stress-rate relation, linear in these rates for the time-rate-independent theory under consideration. The Jaumann stress rate based on the total body spin W is then adopted to ensure compliance of the theory for rigid-body rotation when the strain rate is zero (or equivalently to satisfy the requirement of objectivity). This choice can be considered equivalent to expressing the kinematics relative to axes rotating with the body spin $\mathbf{W}$. The plastic spin, $\mathbf{W}^p$, then becomes $(\mathbf{W}^p - \mathbf{W})$ which is small for small elastic strains ($\mathbf{V}^e \sim \mathbf{I}$) according to (2.14). Thus the last term is (2.13) is small and (2.18) is obeyed approximately. Thus for this choice of axes the assumption of linear kinematics for the strain-rates provides a good approximation but not an exact analysis. For stressing and destressing to the same stress based on fixed axes, as discussed for the standard test described in the Introduction (eqn 1.1 and 1.2), if rotation of the body occurs, the resultant stress increment based on axes having the spin of the body will be non-zero. It is the associated resultant increment of elastic strain which corresponds to the last term in (2.13). Thus the error in the kinematics is largely reduced by selecting an appropriate stress-rate definition.

In the non-linear rate-kinematics approach [2], the process is entirely deductive and leads to a rate-potential stress rate-strain rate relation and thus application of Hill's variational principle valid for finite deformation. It is shown in [2] that for small elastic strains, $\mathbf{V}^e = \mathbf{I} + \boldsymbol{\delta}$, and neglecting $\boldsymbol{\delta}$ relative to $\mathbf{I}$, this theory reduces to that in current use to first order in $\dot{\boldsymbol{\delta}}$ and zero order in $\boldsymbol{\delta}$.

The analysis discussed in this paper has been restricted to isotropic elastic and plastic response to illustrate kinematical aspects with minimum complication arising from the constitutive relation. For satisfactory analysis of deformation at large strains, anisotropy induced by plastic flow must be considered. It is expected that the concept of the unstressed state in providing a convenient structure for the analysis will also be useful for the more general development.

4. A CONFLICTING APPROACH BY S. NEMAT-NASSER[4]

In a recent article [4] Nemat-Nasser claims that the elastic deformation gradient $\mathbf{F}^e$, Fig. 4(b), from the purely plastically deformed unstressed configuration, $\mathbf{p}$, to the configuration, $\mathbf{x}$ (in which stress is applied to the configuration $\mathbf{p}$ which generates purely elastic additional deformation) is not appropriate for expressing the elastic characteristics associated with elastic–plastic deformation. To remedy the alleged "misinterpretations and errors" he devised a new intermediate elastic configuration in which the elastic displacement in the homogeneous deformation case

$$\bar{\mathbf{u}}^e = \mathbf{x} - \mathbf{p} = (\mathbf{F} - \mathbf{F}^p)\mathbf{X} = (\mathbf{F}^e - \mathbf{I})\mathbf{p} \tag{4.1}$$

is considered to be applied to the initially undeformed configuration $\mathbf{X}$. This mapping is expressed by

$$\boldsymbol{\eta}(\mathbf{X}, t) = \mathbf{X} + \bar{\mathbf{u}}^e(\mathbf{X}, t) = \bar{\mathbf{F}}^e\mathbf{X} \tag{4.2}$$

where $\bar{\mathbf{F}}^e$ is the deformation gradient $\partial\boldsymbol{\eta}/\partial\mathbf{X}$. Equation (4.1) corresponds to the relation governing relative displacement of the ends of a vector element of the material:

$$d\bar{\mathbf{u}}^e = d\mathbf{x} - d\mathbf{p} = (\mathbf{F} - \mathbf{F}^p)\, d\mathbf{X} = (\mathbf{F}^e - \mathbf{I})\, d\mathbf{p} \tag{4.3}$$

in the non-homogeneous deformation case considered in the present paper, Fig. 4(b).

Whereas the inclusion of this configuration $\boldsymbol{\eta}$ does generate some simple purely kinematic relations involving a variety of strains and strain rates, one must bear in mind that the objective of the investigation is to generate an elastic–plastic constitutive relation which yields the stress in the elastically–plastically deformed configuration **x** in terms of the history of deformation and this is expressed in terms of its elastic and plastic components because of the physical nature of the phenomena involved. The structure of the constitutive relation and the variables involved must imply appropriate experiments and physical measurements which will express the relevant properties of particular materials. Such considerations were pointed out in Sections 2 and 3 of this paper which emphasized the physically realizable nature of the configurations **X**, **x** and **p** in Fig. 4(b). The commonly discontinuous or multi-valued nature of the mapping **p** from either **X** or **x** is usually avoided in carrying out experimental measurements of material properties by arranging essentially homogeneous deformation so that destressing the body in the deformation $\mathbf{x} \to \mathbf{p}$ is achieved by simply unloading the body since no residual stresses are then generated.

In contrast to **X**, **x** and **p** the configuration $\boldsymbol{\eta}$ comprising the elastic displacements $\bar{u}^e$ from the initially undeformed state is artificial and cannot provide directly information concerning the state of the material in the configuration **x**. The elastic characteristics of the material in the configuration **x** are expressed by the stress variation during the destressing process $\mathbf{x} \to \mathbf{p}$, for example the stress displacement relation along xp during the tension test, Fig. 4(a). The deformation envisaged by Nemat-Nasser, $\mathbf{X} \to \boldsymbol{\eta}$, in which the initially undisturbed configuration is deformed by application of the elastic displacements which would actually occur in the elastic stressing $\mathbf{p} \to \mathbf{x}$, embodies the following characteristics:

(a) it is applied to a body having the elastic and plastic properties of undeformed material, which could differ from those exhibited by the plastically deformed material in the configuration **x**.

(b) since the configuration **X** differs geometrically from **p**, the displacements $\bar{\mathbf{u}}^e$ will produce strains different from the elastic strains $\mathbf{p} \to \mathbf{x}$.

(c) it is likely that elastic deformation $\mathbf{X} \to \boldsymbol{\eta}$ is not achievable even in principle. For example, in the case of the tensile test depicted in Fig. 4(a), the elastic displacement $\bar{u}^e = u(x) - u(p)$ is larger than the initial yield point elastic strain: $U(Y) - u(X)$ due to the work hardening during the deformation $X \to x$ and the reduced slope of the elastic line with increasing plastic flow mentioned in the Introduction. Thus application of the elastic displacement $\bar{u}^e$ to the initial state X will cause plastic flow and the *elastic* state $\boldsymbol{\eta}$ will not be achievable.

Even if such an elastic state could be achieved, conditions (a) and (b) above indicate that the elastic stress generated would not be that acting in the configuration **x** which the constitutive equation we are seeking should predict.

Nemat-Nasser's approach is to devise "elastic and plastic strain rates" using the configurations $\boldsymbol{\eta}$ and **p**. Consider the deformation of a vector element of the material $d\mathbf{X}$ in the undeformed configuration. Using the notation in [4], the relative displacement of its ends is given by

$$d\mathbf{x} - d\mathbf{X} = (\mathbf{F} - \mathbf{I})\, d\mathbf{X} = d\mathbf{U}. \tag{4.4}$$

The relative plastic displacement is given by

$$d\mathbf{p} - d\mathbf{X} = (\mathbf{F}^p - \mathbf{I})\, d\mathbf{X} = d\mathbf{U}^p \tag{4.5}$$

and the relative elastic displacement by

$$d\mathbf{x} - d\mathbf{p} = (\mathbf{F}^e - \mathbf{I})\, d\mathbf{p} = d\boldsymbol{\eta} - d\mathbf{X} = (\bar{\mathbf{F}}^e - \mathbf{I})\, d\mathbf{X} = d\bar{\mathbf{u}}^e. \tag{4.6}$$

Thus

$$d\mathbf{U} = d\bar{\mathbf{u}}^e + d\mathbf{U}^p \tag{4.7}$$

gives

$$\mathbf{F} = \bar{\mathbf{F}}^e + \mathbf{F}^p - \mathbf{I} \tag{4.8}$$

and the velocity gradient in the elastically-plastically deformed configuration is given by

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\bar{\mathbf{F}}}{}^e\mathbf{F}^{-1} + \dot{\mathbf{F}}{}^p\mathbf{F}^{-1} = \hat{\mathbf{L}}^e + \hat{\mathbf{L}}^p. \tag{4.9}$$

Nemat-Nasser *defines* the symmetric parts of the terms on the r.h.s. to be elastic and plastic strain rates, respectively, in the configuration **x**:

$$\mathbf{D} = \hat{\mathbf{D}}^e + \hat{\mathbf{D}}^p. \tag{4.10}$$

This equation is (4.3) of [4] and is used to justify exact summability of elastic and plastic strain rates. It is formally correct, so the question that arises is whether $\hat{\mathbf{D}}^e$ and $\hat{\mathbf{D}}^p$ comprise satisfactory definitions of these strain rates as alternatives to $\mathbf{D}^e$ and $\mathbf{D}^p$ which have been shown in this paper to embody the physical characteristics of elastic and plastic deformation at **x** and to provide significant uncoupling of the phenomena.

To see that $\hat{\mathbf{D}}^e$ is not an appropriate elastic strain rate, consider isotropic ideally plastic material for which plastic flow does not influence the elastic properties. Consider an experiment involving homogeneous irrotational deformation at constant stress at the yield limit, so that plastic flow occurs, but the elastic strains remain constant because the stress and the elastic moduli are constant. Since the shape of the body changes due to plastic flow, the elastic displacements $\bar{\mathbf{u}}^e$ will change in view of the constant elastic strain, $\boldsymbol{\eta}$ will change, hence also $\bar{\mathbf{F}}^e$ and $\hat{\mathbf{D}}^e$ will be non-zero. But the elastic strain is constant!

That such anomalies are bound to arise is clear since by (4.8) and (2.5)

$$\bar{\mathbf{F}}^e = \mathbf{I} + (\mathbf{F}^e - \mathbf{I})\mathbf{F}^p \tag{4.11}$$

so that both $\mathbf{F}^e$ and $\mathbf{F}^p$ contribute to $\bar{\mathbf{F}}^e$ and hence rates of both $\mathbf{F}^e$ and $\mathbf{F}^p$ contribute to $\hat{\mathbf{D}}^e$.

Nemat-Nasser makes the repeated statement that $\mathbf{F}^e$, or the elastic stretch Λ_e in simple tension, does not remain constant when additional infinitesimal purely plastic stretch is applied and implies that this circumstance constitutes an undesirable coupling effect. The concept of coupling can have many interpretations so that great care must be exercised in assessing a particular aspect. For example, if the elastic law is not influenced by plastic flow, the stress is a function of the deformation gradient $\mathbf{F}^e$ only. And yet if a material exhibits work hardening and an increment of plastic tensile strain occurs, the stress must increase and hence necessarily also the elastic strain. The coupling of the elastic strain to the plastic strain which appears to worry Nemat-Nasser is of this nature and, it seems to me, involves no restriction on the use of $\mathbf{F}^e$ to express the elastic state.

In fact his demonstration of this alleged difficulty seems to me to be invalid. Taking $\mathrm{d}S$, $\mathrm{d}s$ and $\mathrm{d}S_p$ to be the lengths of a material element in simple tension in the initial, final and unstressed states respectively (corresponding to $\mathrm{d}\mathbf{X}$, $\mathrm{d}\mathbf{x}$ and $\mathrm{d}\mathbf{p}$ for general deformation), an infinitesimal *purely plastic* deformation which changes $\mathrm{d}s_p$ to $(\mathrm{d}s_p + \epsilon)$, $\epsilon > 0$, is considered. Purely plastic is defined to mean that the elastic *displacement*, $\mathrm{d}s - \mathrm{d}s_p$, remains constant. The elastic stretch ratio, $\Lambda_e = \mathrm{d}s/\mathrm{d}s_p > 1$, thus changes to

$$\frac{\mathrm{d}s + \epsilon}{\mathrm{d}s_p + \epsilon} \simeq \Delta_e - \frac{\epsilon}{\mathrm{d}s_p}(\Lambda_e - 1) < \Lambda_e. \tag{4.12}$$

The reduction in the elastic stretch ratio implies a reduction in stress below yield which is inconsistent with the generation of the postulated plastic displacement increment ϵ. Thus the concept of "purely plastic" deformation which incorporates constant elastic displacement constitutes an invalid thought experiment and no conclusions can be drawn from it. Nemat-Nasser draws the remarkable conclusion that Λ_e is an inappropriate measure of elastic stretch ([4], p. 158). Surely, given an unstressed piece of material (irrespective of its previous history of deformation) which exhibits elastic response to applied stress, one unequivocal characteristic is that the deformation gradient $\mathbf{F}^e$ from that unstressed state (or equivalently the stretch ratio Λ_e for simple tension) is an appropriate variable with which to express the elastic response! The usual finite-deformation elastic law, (2.15), for stress as a function of $\mathbf{F}^e$ is adopted in [2] and since it is an instantaneous function relation the kinematical question of rates of strain does not arise. Both sides of the equality are then acted on by the same differential operators, thus yielding further identities, and the process is, as Eddington states, just mathematics. This leads to an equation involving a Jaumann derivative of the Kirchhoff stress τ and the rate of deformation $\mathbf{D}^e$, which is directly substituted into the kinematic relation (2.13) to eliminate $\mathbf{D}^e$.

It seems to me that a problem with the analysis in [4], e.g. the interpretation of (4.12) above is the use of the elastic *displacement* as a variable with which to investigate elastic–plastic deformation when it is strains and deformation gradients which are appropriate. It also seems to me that transfer of the elastic displacement vector $\bar{\mathbf{u}}^e$ from the base configuration $\mathbf{p}$ to the initial configuration $\mathbf{X}$ to yield the configuration η similarly causes the difficulties of interpretation already discussed.

Quite apart from the difficulties already discussed of interpreting elastic–plastic kinematics in terms of the mapping $\eta(\mathbf{X}, t)$, in which the actual elastic deformation from $\mathbf{p}$ to $\mathbf{x}$, Fig. 4(b), is superimposed on the initial configuration $\mathbf{X}$, one must address the question of the purpose of that kinematical exercise. We need to be able to predict the stress variation as a functional of the deformation history along a path such as $\mathbf{X}\rightarrow\mathbf{x}$ in Fig. 4(b) along which elastic–plastic deformation is taking place. In particular we will consider determining the stress in the configuration $\mathbf{x}$. In the fomulation of such an elastic–plastic constitutive relation, as, for example, developed in [2, 15], since the plastic law is of flow or equivalently of incremental type, a kinematical relation expressing the total strain rate at the configuration $\mathbf{x}$ in terms of elastic and plastic components must first be developed or hypothesized. Then the elastic and plastic constitutive relations are used to eliminate the elastic and plastic strain rates in favor of stress, stress-rate, etc. at the configuration $\mathbf{x}$. This then yields an expression for the stress-rate at $\mathbf{x}$ in terms of the deformation history $\mathbf{X}\rightarrow\mathbf{x}$ (not involving the configuration $\mathbf{p}$) which can be integrated to give the stress variation. When $\mathbf{D}^e$ and $\mathbf{D}^p$ are used as the elastic and plastic strain rate variables, the derivative of the elasticity law (2.15), or of Hooke's law in the linear elastic case, expresses $\mathbf{D}^e$ in terms of the stress-rate at $\mathbf{x}$, and $\mathbf{D}^p$ is given by the plasticity law. However, if $\hat{\mathbf{D}}^e$, associated with the configuration η, is used as the elastic strain-rate variable, for the reasons already discussed that the geometry, elastic characteristics and yield condition along $\eta(\mathbf{X}, t)$ are not directly related to those on the elastic loading path $\mathbf{p}\rightarrow\mathbf{x}$, no expression for the stress rate at $\mathbf{x}$ can be deduced to eliminate $\hat{\mathbf{D}}^e$. Thus the use of $\hat{\mathbf{D}}^e$ does not provide a building block for determining a constitutive relation involving the stress and total strain rate $\mathbf{D}$ along the actual deformation path $\mathbf{X}\rightarrow\mathbf{x}$, this quite apart from whether $\hat{\mathbf{D}}^e$ is a satisfactory variable with which to express elastic strain-rate.

Kinetic (i.e. force related) matters in addition to kinematic questions are addressed in [4], specifically the separation of rate of work or power into elastic and plastic components. The stress in the state $\mathbf{x}$ is utilized without comment concerning its relation, or lack of one, to $\hat{\mathbf{D}}^e$. Since $\mathbf{D}$ is separated into additive components: $\hat{\mathbf{D}}^e$ and $\hat{\mathbf{D}}^p$, the sum of the "plastic" and "elastic" powers is equal to the total power as would any arbitrary division of $\mathbf{D}$ into additive components generate. Thus the brief kinetic discussion adds little to elucidating the formulation of the elastic–plastic constitutive relation.

There is one seeming inconsistency which might have appeared evident in this article, that in spite of the stress placed on physical realizability of concepts, the definition of an artificial plastic strain was suggested in Section 3 for the case when reversed plastic flow (the Bauschinger effect) occurred during destressing. Since destressing is then not purely elastic, the plastic strain in the unstressed configuration $\mathbf{p}$ is not the same as that prior to unloading at $\mathbf{x}$. Thus the strain at $\mathbf{p}$, which is purely plastic since the stress and hence elastic strain is zero, does not provide a measure of the plastic strain at $\mathbf{x}$. Elastic destressing from $\mathbf{x}$ is governed by a free-energy function but the range over which this is valid (inside the current yield surface) does not extend to zero stress. The assumption that the elastic law applies to zero stress, so that no plastic flow is permitted, yields a measure of the plastic strain at $\mathbf{x}$, a kind of analytic extension of the theory presented in this paper, needed to define a unique plastic strain at $\mathbf{x}$ when a Bauschinger effect intervenes in the elastic unloading to zero stress. In principle this strain cannot be measured but may be of some value in interpreting certain situations. It is not *needed* in the evaluation of any physically achievable configuration and so is not at odds with the approach to physical problems expressed in this paper.

5. CONCLUDING REMARKS

It seems to me that the analysis of elastic-plastic deformation at finite strain poses some challengeing problems to incorporate the influence of plastic flow on elastic characteristics and vice versa. It seems to me that the kinematical questions mainly addressed in this paper comprise a relatively clear and simple aspect of the problem and I trust that we, the mechanics community, can apply ourselves to the more general questions. When we consider deformations beyond moderate strains, both plastic and elastic anisotropy induced by plastic flow will have to be considered and this will introduce new concepts. For example, the elastic strain increment associated with adding a stress increment and then removing it will not be reversible because on the loading part of the cycle changes in elastic characteristics will be occurring due to the accompanying increment of plastic strain. This will add another aspect to the circumstance that the residual part of the strain increment is not the plastic strain increment following application and removal of a stress increment.

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